Dynamical Systems

Exam 22 January 2024

university of groningen

The exam consists of 4 questions. You have 120 minutes to do the exam. You can achieve 50 points in total which includes a bonus of 5 points.

1. [**3+3+3=9 Points**] For each of the following time-continuous systems that depend on a parameter $a \in \mathbb{R}$, sketch the bifurcation diagram including representative phase portraits and classify the bifurcations of equilibrium points.

   (a) The one-dimensional systems $x' = ax - x^2$.

   (b) The one-dimensional systems $x' = x\cos x + ax$.

   (c) The planar systems

   $$\begin{aligned} r' &= r - r^3, \\ \theta' &= a + \sin\theta, \end{aligned}$$

   where r and θ are polar coordinates. In this case sketch representative phase portraits in the Cartesian coordinate plane and sketch the bifurcation diagram in a diagram θ versus a.

2. [**9 Points**] Consider the planar systems

   $$X' = \begin{pmatrix} a & b \\ 1 & a \end{pmatrix} X$$

   with parameters $a, b \in \mathbb{R}$. Sketch the regions in the (a, b) plane where this system has different types of canonical forms. In each region give the canonical form and sketch the phase portrait of the system in canonical form.

3. [**4+3+4+2=13 Points**] Consider the planar system

   $$\begin{aligned} x' &= y, \\ y' &= -\nu y + x^2 - 1, \end{aligned}$$

   where $\nu \geq 0$ is a parameter.

   (a) Show that the system has the two equilibrium points $(x_-, y_-) = (-1, 0)$ and $(x_+, y_+) = (1, 0)$, and determine their stability from the linearization.

   (b) Show that for $\nu = 0$, the system is Hamiltonian with Hamilton function

   $$H(x, y) = \frac{1}{2}y^2 - \frac{1}{3}x^3 + x + \frac{2}{3}$$

   and sketch the phase portrait in the (x, y) plane.

– **please turn over** –

(c) Show that for $\nu \geq 0$ and each $0 < h < 4/3$, H is a Lyapunov function in the region $D_h = \{(x, y) \in \mathbb{R}^2 \mid H(x, y) \leq h,\ x < 1\}$ and use the Lasalle Invariance Principle to show that for $\nu > 0$, the equilibrium at $(x_-, y_-) = (-1, 0)$ is asymptotically stable with D_h belonging to the basin of attraction.

(d) Sketch the phase portrait for $\nu > 0$ by paying attention to the stable and unstable curves of the saddle at $(x_+, y_+) = (1, 0)$. What can you say about the basin of attraction of $(x_-, y_-) = (-1, 0)$?

4. **[3+9+2=14 Points]** Consider the one-dimensional discrete-time system $x_{n+1} = 10x_n \bmod 1$, $n = 0, 1, 2, \ldots$, on the interval $[0, 1]$. Note that upon writing x_n in decimal form the system is described by the map $0.d_1d_2d_3d_4\ldots \mapsto 0.d_2d_3d_4\ldots$.

(a) Show that for any positive integer p, the system has a periodic orbit of (minimal) period p and show that all such periodic orbits are unstable.

(b) Show that the system satisfies all three conditions of Devaney's definition of chaos.

(c) Show that the system has uncountably many non-periodic points.